Calculus 1
Midterm Exam – Solutions
September 30, 2026 (18:30 – 20:30)

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1) Prove using the (ε, δ)-definition that $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

Solution. Let $\varepsilon > 0$ be arbitrary and take $\delta = \min\{a, \varepsilon\sqrt{a}\}$. Then, $0 < |x - a| < \delta$ implies that $x > 0$ and

$$|\sqrt{x} - \sqrt{a}| = \left|(\sqrt{x} - \sqrt{a})\frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right| = \left|\frac{|x| - |a|}{\sqrt{x} + \sqrt{a}}\right| < \left|\frac{x - a}{\sqrt{a}}\right| = \frac{|x - a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} \leq \varepsilon.$$

Thus, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

2) Apply l'Hospital's rule to evaluate the following limit: $\lim_{x\to\infty} \left[x - x^2 \ln\left(\frac{1+x}{x}\right)\right]$. Indicate which rules of differentiation are being used in each step.

Solution. The limit in question has an indeterminate form of type "$\infty - \infty$". Let us write

$$L = \lim_{x\to\infty} \left[x - x^2 \ln\left(\frac{1+x}{x}\right)\right] = \lim_{x\to\infty} \left[x - x^2 \ln\left(\frac{1}{x} + 1\right)\right]$$

and let $t = 1/x$, so as $x \to \infty$, we have $t \to 0^+$. This change of variable yields

$$L = \lim_{t\to 0+} \left[\frac{1}{t} - \frac{1}{t^2}\ln(t+1)\right] = \lim_{t\to 0+} \frac{t - \ln(t+1)}{t^2},$$

which has an indeterminate form as desired, specifically "$0/0$". Now we can use l'Hospital's rule to obtain

$$L = \lim_{t\to 0+} \frac{t - \ln(t+1)}{t^2} \overset{\text{l'H}}{=} \lim_{t\to 0+} \frac{1 - \frac{1}{t+1}}{2t} = \lim_{t\to 0+} \frac{t/(t+1)}{2t} = \lim_{t\to 0+} \frac{1}{2(t+1)} = \frac{1}{2}$$

Above we made use of the Difference Rule, the Basic Derivative $(t)' = 1$, the Chain Rule, the derivative $(\ln x)' = 1/x$, and the Power Rule $(t^2)' = 2x$. The last equality follows by direct substitution.

Note: Starting the solution by factoring out x or x^2 leads to a more complicated solution. Any complete solution is awarded full points as long as l'Hospital's rule is correctly applied and the rules of differentiation are well referenced.

3) Use differentiation to find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

Solution. The distance between the points (x, y) and $(1, 0)$ is given by

$$d = \sqrt{(x-1)^2 + y^2} \quad \Leftrightarrow \quad d^2 = (x-1)^2 + y^2.$$

We need to find the maximum of d whilst simultaneously making sure that $4x^2 + y^2 = 4$. We can make things simpler by looking for the maximum of d^2 instead. This is equivalent to the original problem since the square (root) function is strictly increasing over the positive half-line $(0, \infty)$.

For points (x, y) on the ellipse we have $4x^2 + y^2 = 4$, that is $y^2 = 4 - 4x^2$. We substitute this into d^2 to turn it into a function of x:

$$f(x) = (x-1)^2 + (4 - 4x^2), \quad x \in [-1, 1].$$

Note that the domain of f is the closed interval $[-1, 1]$. For x outside this interval, the equation $4x^2 + y^2 = 4$ has no solutions.

To find the maximum, we locate the critical values. Since $f(x) = 5 - 2x - 3x^2$ is a polynomial in x, it is differentiable everywhere. Therefore the critical points are the solutions of the equation $f'(x) = 0$. By the Sum, Constant Multiple, and Power Rules of Differentiation, we have $f'(x) = -2 - 6x = 0$ if and only if $x = -1/3$. Thus the only critical point for f is at $x = -1/3$. There it has a (local) maximum. This can be shown via the First Derivative Test as f' changes sign from positive to negative as x increases through $-1/3$. Alternatively, one may use the Second Derivative Test and obtain $f''(-1/3) = -6 < 0$. At $x = -1/3$ the function attains the (local) maximum value $f(-1/3) = 16/3$. To see whether this is an absolute maximum, we need to use the Closed Interval Method and compare it to the end-point values, namely $f(-1) = 4$ and $f(1) = 0$. And indeed, we find that $f(-1/3) = 16/3$ is the greatest of these values. Therefore we can conclude that the squared-distance function around the ellipse $f(x)$ attains its absolute maximum at $x = -1/3$. The corresponding y-coordinates are obtained from the equations of the ellipse. We find $y = \pm 4\sqrt{2}/3$.

In conclusion, there are two points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$, they are are $(-1/3, 4\sqrt{2}/3)$ and $(-1/3, -4\sqrt{2}/3)$.

4) Suppose that $f'(x) \geq 3$ for all $x \in \mathbb{R}$. Show that $f(2) + 18 \leq f(8)$.

Solution. The function f is continuous and differentiable everywhere since differentiability implies continuity and we are told that $f'(x)$ exists for all x. In particular, it is also continuous on the closed interval $[2, 8]$ and differentiable on the open interval $(2, 8)$. By the Mean Value Theorem, there exists a $c \in (2, 8)$ such that

$$f'(c) = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - f(2)}{6}.$$

But we are told that $f'(x) \geq 3$ for all x, thus also for $x = c$, hence we have

$$3 \leq f'(c) = \frac{f(8) - f(2)}{6},$$

or equivalently,

$$18 \leq f(8) - f(2) \quad \Leftrightarrow \quad f(2) + 18 \leq f(8).$$

This concludes the proof. An alternative solution involves integrating the inequality $f'(x) \geq 3$ with respect to x, and evaluating the resulting inequality at $x = 2$ and at $x = 8$.

5) Evaluate the indefinite integral $\int \sqrt{1 - x^4}\, dx$ as an infinite series.

Solution. Recall the Binomial Series

$$(1 + u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k, \quad \text{where } \binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha - 1)\dots(\alpha - k + 1)}{k!} \text{ for } k = 1, 2, 3, \dots$$

and the radius of convergence is $R = 1$. Now substitute $\alpha = 1/2$ and $u = -x^4$ above to get

$$\sqrt{1 - x^4} = (1 - x^4)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-x^4)^k = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} x^{4k} = 1 - \frac{x^4}{2} - \frac{x^8}{8} - \frac{x^{12}}{16} - \dots$$

Now for $|x| < R = 1$ we may integrate term-by-term (see Theorem 2 on page 789 of the Textbook) resulting in

$$\int \sqrt{1 - x^4}\, dx = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \int x^{4k}\, dx = C + \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{x^{4k+1}}{4k + 1},$$

where we used the power rule. The coefficients may be expressed using the factorial (and double factorial) as follows:

$$\frac{(-1)^k \binom{1/2}{k}}{4k + 1} = -\frac{(2k - 3)!!}{k! 2^k (4k + 1)}.$$

6) Prove the following reduction formula for all $n \in \mathbb{N}$: $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$. (Hint: Use integration by parts.)

Solution. We give a direct proof. Following the hint, we apply integration by parts

$$\int u \, dv = uv - \int v \, du$$

with $u = (\ln x)^n$ and $dv = dx$. The Chain and Power Rules of Differentiation as well as the derivative $(\ln x)' = 1/x$ yield $du = n(\ln x)^{n-1}(1/x)dx$ and $v = x$. Therefore we have

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int xn(\ln x)^{n-1}(1/x)dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx.$$

Above we used the constant multiple rule. Note that the all steps are valid for any $n \in \mathbb{N}$.